OPTIMAL BOUNDS FOR SELF-INTERSECTION LOCAL TIMES

GEORGE DELIGIANNIDIS AND SERGEY UTEV

Abstract. For a random walk $S_n, n \geq 0$ in $\mathbb{Z}^d$, let $l(n,x)$ be its local time at the site $x \in \mathbb{Z}^d$. Define the α-fold self intersection local time $L_n(\alpha) := \sum_x l(n,x)^\alpha$, and let $L_n(\alpha|\epsilon, d)$ the corresponding quantity for d-dimensional simple random walk. Without imposing any moment conditions, we show that the variances of the local times $\mathrm{var}(L_n(\alpha))$ of any genuinely d-dimensional random walk are bounded above by the corresponding characteristics of the simple symmetric random walk in $\mathbb{Z}^d$, i.e. $\mathrm{var}(L_n(\alpha)) \leq C \,\mathrm{var}[L_n(\alpha|\epsilon, d)] \sim K_{d,\alpha} v_{d,\alpha}(n)$. In particular, variances of local times of all genuinely d-dimensional random walks, $d \geq 4$, are similar to the 4-dimensional symmetric case $\mathrm{var}(L_n(\alpha)) = O(n)$. On the other hand, in dimensions $d \leq 3$ the resemblance to the simple random walk $\liminf_{n\to\infty} \mathrm{var}(L_n(\alpha))/v_{d,\alpha}(n) > 0$ implies that the jumps must have zero mean and finite second moment.

1. Introduction and main results

Let $X, X_1, X_2, \dots$ be independent, identically distributed, $\mathbb{Z}^d$-valued random variables, and define the random walk $S_0 := 0$, $S_n = \sum_{j=1}^n X_j$, for $n \geq 1$ Let $l(n,x) = \sum_{j=1}^n \mathbf{I}(S_j = x)$ be the local time of $(S_n)_n$ at the site $x \in \mathbb{Z}^d$, and define for a positive integer α the α-fold *self-intersection local time*

$$L_n = L_n(\alpha) = \sum_{x\in\mathbb{Z}^d} l(n,x)^\alpha = \sum_{i_1,\dots,i_\alpha=0}^{n} \mathbf{I}(S_{i_1} = \cdots = S_{i_\alpha}).$$

Our method also applies to the more general case where the X_i are independent but not identically distributed. To distinguish between the two cases, we shall refer to random walk with independent identically distributed increments as the i.i.d. case. Following Spitzer [19], in the i.i.d. case, we call X_i and the random walk it generates *genuinely d-dimensional* if the support of the variable $X_1 - X_2$ linearly generates d-dimensional space. Finally let $\Gamma = [0, 2\pi]^d$.

The quantity $L_n(\alpha)$ has received considerable attention in the literature due to its relation to *self-avoiding walks* and *random walks in random scenery*. In particular let the *random scenery* $\{\xi_x, x \in \mathbb{Z}^d\}$ be a collection of i.i.d. random variables, independent of the X_i, and define the process $Z_0 = 0$, $Z_n = \sum_{i=1}^n \xi_{S_i}$. Then $(Z_n)_n$ is commonly referred to as *random walk in random scenery* and was introduced in Kesten and Spitzer [13], where functional limit theorems were obtained for $Z_{[nt]}$ under an appropriate normalization for the case $d = 1$. The case $d = 2$, with X_i centered with non-singular covariance matrix, was treated in [4] where it was shown that $Z_{[nt]}/\sqrt{n \log n}$ converges weakly to Brownian motion. As is obvious from the identities $Z_n = \sum_{x\in\mathbb{Z}^d} l(n,x)\xi_x$, $\mathrm{var}(Z_n) = \mathrm{var}[L_n(2)]\,\mathrm{var}(\xi_x)$, limit theorems for Z_n usually require asymptotics for the local times of the random walk $(S_n)_n$.

Such asymptotics are usually obtained from Fourier techniques applied to the characteristic function $f(t) = \mathbb{E}[\exp(it\cdot X)]$ under the additional assumption of a Taylor expansion of the form $f(t) = 1 - \langle \Sigma t, t\rangle + o(|t|^2)$ where Σ is the positive definite covariance matrix [4, 5, 6, 12, 20], which further requires that $\mathbb{E}|X|^2 < \infty$ and $\mathbb{E}X = 0$. Similar restrictions are also required for the application of local limit theorems such as in [14, 17].

In this paper, motivated by the results of Spitzer [19] for genuinely d-dimensional random walks and the approach of Becker and König [3](see also Asselah [2] where non-integer α is also treated) we shall study the asymptotic behavior of $\mathrm{var}(L_n(\alpha))$ without imposing any moment

2000 *Mathematics Subject Classification.* Primary 60G50, 60F05.

Key words and phrases. Self-intersection local time, random walk in random scenery.

assumptions on the random walk. The central idea behind our approach is to compare the self-intersection local times $L_n(\alpha)$ of a general d-dimensional walk with those of its symmetrised version. In addition we will compare the self-intersection local times of a general d-dimensional random walk with those of the d-dimensional simple symmetric random walk, $S_n^{\epsilon,d}$ which we denote by $L_n(\alpha|\epsilon, d)$. Recall that simple random walk in $\mathbb{Z}^d$ is defined as $S_0^{\epsilon,d} := 0$, $S_n^{\epsilon,d} := \sum_{j=1}^n X_j^{\epsilon,d}$ for $n \geq 1$, where for $k = 1, \ldots, d$ $\mathbb{P}(X_j^{\epsilon,d} = \pm e_k) = 1/2d$ and e_k is the k-th unit coordinate vector. It is well-known that with some positive constant $K_{\alpha,d}$, $\text{var}[L_n(\alpha|\epsilon, d)] \sim K_{\alpha,d} v_{d,\alpha}(n)$ where

$$v_{1,\alpha}(n) = n^{1+\alpha}, \quad v_{2,\alpha}(n) = n^2 \log(n)^{2\alpha-4}, \quad v_{3,\alpha}(n) = n \log(n) \quad \text{and} \quad v_{d,\alpha}(n) = n, \quad d \geq 4.$$

Several other cases have been treated in the literature, using a variety of methods.

A careful look at the literature reveals that the most difficult case in $d = 2$ is the *near transient recurrent* case, where $\mathbb{P}(S_n = 0) \sim C/n$, which corresponds to genuinely 2-dimensional symmetric recurrent random walks, which will be referred to as a critical case. Surprisingly enough, the variance of the self-intersection local times in the critical case is asymptotically the largest.

Theorem 1. *Let X_i be i.i.d., genuinely d-dimensional. Then,*

$$\text{var}(L_n(\alpha)) \leq c_{\alpha,X} \, \text{var}\left(L_n(\alpha|\epsilon, d)\right) \leq C_{\alpha,X} v_{d,\alpha}(n) \, .$$

The result was motivated by [19] and [3] (and improves related results of Becker and Konig for $d = 3$ and $d = 4$). Several cases treated in [2, 4, 5, 8, 10, 7, 3, 17] can then be obtained as particular cases.

Moreover, we also show the surprising reverse, more exactly that the right asymptotic of $\text{var}(L_n)$ implies that the jumps must have zero mean and finite second moment.

Theorem 2. *Let X_i be i.i.d., genuinely d-dimensional and $d = 1, 2, 3$. If*

$$\liminf_{n\to\infty} \frac{\text{var}(L_n(\alpha))}{v_{d,\alpha}(n)} > 0,$$

then $\mathbb{E}|X|^2 < \infty$ and $\mathbb{E}X = 0$.

As it follows from Theorem 3, given below, for $d = 2, 3$ and Theorem 5.2.3 in Chen [8] for $d = 1$, if $\mathbb{E}X = 0$ and $0 < \mathbb{E}|X|^2 < \infty$, then $\liminf_n \text{var}[L_n(\alpha)]/v_{d,\alpha}(n) > 0$.

For general genuinely d-dimensional random walks with finite second moments and zero mean, the asymptotic behavior is similar to d-dimensional simple symmetric random walk, again the most complicated case being $d = 2$. Also, as it follows from our general bounds (see Proposition 4 and Corollary 7), the asymptotics for the genuinely d-dimensional random walk can be reproduced by those of the symmetric one-dimensional random walk with appropriately chosen heavy tails, as was indicated by Kesten and Spitzer [13]. The proofs are based on adapting the Tauberian approach developed in [10].

Theorem 3. *Let $d = 1, 2, 3$, and suppose that for $t \in [-\pi, \pi]^d$ we have*

$$f(t) = 1 - \gamma|t| + R(t), \text{ for } d = 1, \text{ or } \qquad f(t) = 1 - \langle \Sigma t, t\rangle + R(t), \text{ for } d = 2, 3, \tag{1}$$

where Σ is a non-singular covariance matrix and $R(t) = o(|t|)$ for $d = 1$, and $o(|t|^2)$ for $d = 2, 3$ as $t \to 0$. Then

$$\text{var}(L_n(\alpha)) \sim \begin{cases} \frac{(\pi^2+6)}{12} \frac{(\alpha!)^2(\alpha-1)^2}{(\gamma\pi)^{2\alpha-2}} n^2 \log(n)^{2\alpha-4}, & \text{for } d = 1, \\ \frac{(\alpha!)^2(\alpha-1)^2}{2\left(2\pi\sqrt{|\Sigma|}\right)^{2\alpha-2}} n^2 \log(n)^{2\alpha-4}(\kappa+1) & \text{for } d = 2, \text{ and} \\ (\kappa_1 + \kappa_2) n \log n, & \text{for } d = 3, \ \alpha = 2, \end{cases}$$

where $\kappa = \int_0^\infty \int_0^\infty \mathrm{d}r\mathrm{d}s \left[(1 + r)(1 + s)\sqrt{(1 + r + s)^2 - 4rs}\right]^{-1} - \pi^2/6$ and κ_1, κ_2 are defined in (7) and (9) respectively.

Moreover, if $L'(n, \alpha)$ is the self-intersection local time of another random walk whose characteristic function also satisfies (1) then $L'(n, \alpha) = L(n, \alpha)(1 + o(1))$.

The methods developed in this paper are used by the first author and K. Zemer in [11] to prove that the range of 1-stable random walk in $\mathbb{Z}$ and simple random walk in $\mathbb{Z}^2$ has the Fölner property and therefore to compute the relative complexity of random walk in random scenery in the sense of Aaronson [1].

2. Proofs

2.1. **General bounds.** We first develop a technique to treat random walks with independent but not necessarily identically distributed increments.

Proposition 4. (General upper bound) *Assume that X_i are independent $\mathbb{Z}^d$-valued random variables and let $S_{u,v} := X_u + \ldots + X_{u+v}$. Suppose further that for all $n \in \mathbb{N}$, and integers $a, u, b, v \geq 0$, with $a + u \leq b$ and any $x \in \mathbb{Z}^d$ we have*

$$\mathbb{P}\Big(S_{a,u} \pm S_{b,v} = x\Big) \leq \phi(u+v), \tag{A}$$

$$\mathbb{P}\Big(S_{a,u} = 0\Big) - \mathbb{P}\Big(S_{a,u} + S_{b,v} = 0\Big) \leq \psi(u,v), \tag{B}$$

where $\phi(u)$ is non-increasing, $\psi(u,v)$ is non-increasing in u and is non-decreasing and sub-additive in v in the sense that $\psi(u, v+w) \leq A_\psi[\psi(u,v) + \psi(u,w)]$, for some constant A_ψ independent of u, v, w. Then, for some constant $K = cA_\psi(1+A_\psi)^{\alpha-2}$ depending only on α

$$\mathrm{var}(L_n(\alpha)) \leq Kn\Big(\sum_{i=0}^{n-1}\phi(i)\Big)^{2\alpha-4}\sum_{i,j,k=0}^{n-1}\big[\phi(j\vee i)\phi(k\vee i) + \phi(j)\psi(i+k,j)\big].$$

Proof of Proposition 4. We first write out the variance as a sum

$$\begin{aligned}\mathrm{var}\, L_n(\alpha) = (\alpha!)^2 \sum_{k_1\leq\cdots\leq k_\alpha}\sum_{l_1\leq\cdots\leq l_\alpha}\Big(&\mathbb{P}[S_{k_1} = \cdots = S_{k_\alpha}, S_{l_1} = \cdots = S_{l_\alpha}]\\ &- \mathbb{P}[S_{k_1} = \cdots = S_{k_\alpha}]\mathbb{P}[S_{l_1} = \cdots S_{l_\alpha}]\Big).\end{aligned} \tag{2}$$

An important role is played by the manner in which the two sequences are interlaced, since for example if $k_\alpha \leq l_1$ or $l_\alpha \leq k_1$, the term vanishes by the Markov property. Let's assume, without loss of generality, that $k_1 \leq l_1$ and we arrange the two sequences in an ordered sequence of combined length 2α which we denote as $(p_1, \ldots, p_{2\alpha})$; we also define $(\epsilon_1, \ldots, \epsilon_{2\alpha})$ where $\epsilon_i = 0$ if p_i came from $\mathbf{k} := \{k_1, \ldots, k_\alpha\}$, and $\epsilon_i = 1$ if p_i came from $\mathbf{l} := \{l_1, \ldots, l_\alpha\}$. Finally we define two new sequences $m_0, m_1, \ldots, m_{2\alpha-1}$, and $\delta_1, \ldots, \delta_{2\alpha-1}$, where $m_0 := p_1$, $m_i = p_{i+1} - p_i$ and $\delta_i = \epsilon_{i+1} - \epsilon_i$, for $i = 1, \ldots, 2\alpha - 1$. Notice that since we assume that $k_1 \leq l_1$, we have $p_1 = k_1$ and $\epsilon_1 = 0$. Let $v(\delta) := \sum_{i=1}^{2\alpha-1}|\delta|$, denote the *interlacement index*. The terms with $v = 1$ vanish, while the terms with $v = 2$ will be considered separately.

We first consider the sum I_n of the terms with $v \geq 3$ for which we drop the negative part and sum over the free index $m_0 = k_1$ to obtain the bound

$$I_n \leq c(\alpha)n \sum_{m_1,\ldots,m_{2\alpha-1}}\sum_{x\in\mathbb{Z}^d}\prod_{t=1}^{2\alpha-1}\sup_w \mathbb{P}(S_{w,m_t} = \delta_t x),$$

where $c(\alpha)$ denotes generic constants depending only on α, which may change from line to line. Of these $2\alpha - 1$ δ's, exactly $u := 2\alpha - 1 - v$ are equal to 0, and therefore

$$I_n \leq c(\alpha)n\Big[\sum_{i=0}^{n}\phi(i)\Big]^u \sum_{j_1,\ldots,j_v=0}^{n}\sum_x\prod_{t=1}^{v}\sup_{w_1,\ldots,w_v}\mathbb{P}(S_{w_t,j_t} = \delta_t x).$$

Notice that if $S^{(1)}, \ldots, S^{(v)}$ denote independent random walks then, assuming without loss of generality that $j_1 \leq \cdots \leq j_v$, we have that

$$\sum_x\prod_{t=1}^{v}\mathbb{P}(S_{w_t,j_t} = \delta_t x) \leq \Big(\prod_{t=2}^{v-1}\max_x \mathbb{P}(S^{(t)}_{j_t} = x)\Big)\mathbb{P}(S^{(1)}_{j_1} = \delta_v S^{(v)}_{j_v}) \tag{3}$$

$$\leq \phi(j_1 + j_v) \prod_{t=2}^{v-1} \phi(j_t) \leq \prod_{t=2}^{v} \phi(j_t \vee j_1).$$

Writing $G_n := \sum_{i=0}^{n} \phi(i)$, since ϕ is non-increasing we have that

$$\Delta_{n,v} := \sum_{0\leq j_1 \leq \cdots \leq j_v \leq n} \prod_{t=2}^{v} \phi(j_t \vee j_1) \leq \sum_{j_v=0}^{n} \phi(j_v) \sum_{0\leq j_1\leq\cdots\leq j_{v-1}\leq n} \prod_{t=2}^{v-1} \phi(j_t \vee j_1) = G_n \Delta_{n,v-1},$$

and repeating this procedure, for $v \geq 3$ we have that $\Delta_{n,v} \leq \Delta_{n,3} G_n^{v-3}$. Combining the two bounds and summing over $v = 3, \ldots, 2\alpha - 1$, we have the upper bound

$$\sum_{v=3}^{2\alpha-1} c(\alpha) n G_n^{2\alpha-1-v} \Delta_{n,v} \leq c(\alpha) n G_n^{2\alpha-1-v+v-3} \Delta_{n,3} = c(\alpha) n G_n^{2\alpha-4} \Delta_{n,3}.$$

Next we consider the sum J_n over the terms with $v = 2$, which occurs when for some j, the indices $l_1, \ldots, l_\alpha$ all lie in $[k_j, k_{j+1}]$. Then it is easy to see that this sum J_n is bounded above by

$$\begin{aligned}
J_n \leq c(\alpha) n &\sup_{w_0,\ldots,w_{2\alpha-1}} \sum_{m_{\alpha+1},\cdots,m_{2\alpha-2}=0}^{n} \prod_{r=\alpha+1}^{2\alpha-2} \mathbb{P}(S_{w_r,m_r} = 0) \\
&\times \sum_{m_0,\cdots,m_\alpha=0}^{n} \left[\prod_{t=1}^{\alpha-1} \mathbb{P}(S_{w_t,m_t} = 0) \right] \Big[\mathbb{P}(S_{w_0,m_0} + S_{w_\alpha,m_\alpha} = 0) - \mathbb{P}(S_{w_0,m_0} + \ldots + S_{w_\alpha,m_\alpha} = 0) \Big] \\
&\leq c(\alpha) n G_n^{\alpha-2} \sup_{w_0,\ldots,w_\alpha} \\
&\times \sum_{m_0,\cdots,m_\alpha=0}^{n} \left[\prod_{t=1}^{\alpha-1} \mathbb{P}(S_{w_t,m_t} = 0) \right] \Big[\mathbb{P}(S_{w_0,m_0} + S_{w_\alpha,m_\alpha} = 0) - \mathbb{P}(S_{w_0,m_0} + \ldots + S_{w_\alpha,m_\alpha} = 0) \Big] \\
&\leq c(\alpha) n G_n^{\alpha-2} \sum_{m_0,\cdots,m_\alpha=0}^{n} \left[\prod_{t=1}^{\alpha-1} \phi(m_t) \right] \psi(m_0 + m_\alpha, m_1 + \cdots + m_{\alpha-1}) \\
&\leq c(\alpha) n G_n^{\alpha-2} A_\psi (1 + A_\psi)^{\alpha-2} \left(\sum_{m_2,\cdots,m_{\alpha-1}} \prod_{t=2}^{\alpha-1} \phi(m_t) \right) \times \sum_{m_0,m_1,m_\alpha} \phi(m_1) \psi(m_0 + m_\alpha, m_1) \\
&\leq c(\alpha) A_\psi (1 + A_\psi)^{\alpha-2} n G_n^{2\alpha-4} \sum_{i,j,k=0}^{n} \phi(j) \psi(i + k, j). \qquad \square
\end{aligned}$$

The following corollary provides explicit bounds in the cases that are usually considered in the literature.

Corollary 5. *Assume that the conditions of Proposition 4 are satisfied with $\phi(m) = Tm^{-r}$ and $\psi(m,k) = Tm^{-r-1}(k \wedge m)$. Then,*

$$\operatorname{var}(L_n(\alpha)) \leq c(\alpha) T^{2\alpha-2} \times \begin{cases} n^2 \log(n)^{2\alpha-4}, & \text{if } r = 1, \\ n^{4-2r}, & \text{if } 1 < r < 3/2, \\ n \log(n), & \text{if } r = 3/2, \text{ and} \\ n, & \text{if } r > 3/2. \end{cases}$$

Several relevant results treated so far in [3, 4, 7, 20, 8, 10, 14, 17] are not only obtained as a special case but also extended to the case of independent but not necessarily identically distributed variables, for example by applying the local limit theorem, as it is conducted in [14].

Also when X_i is in the domain of attraction of the one-dimensional symmetric Cauchy law ([9, 10]), or in the case of strongly aperiodic planar random walk with second moments ([4, 7, 20, 14, 17]), it is well known that the conditions of Proposition 4 are satisfied with $\phi(m) = T/m$ and $\psi(m,k) = Tm^{-2}(k \wedge m)$.

However, we can do better for symmetrized variables and show that condition (A) implies (B), which together with the comparison technique motivate the following results.

Proposition 6 (Bound via comparison with symmetrised)**.** *Let X_i be independent, d-dimensional random variables and $f_i(t) := \mathbb{E}\exp(\mathrm{i}tX_i)$, and assume that there exists a non-negative measurable function $f(t)$, $0 \le f(t) \le 1$ and positive non-increasing sequence $\phi(m)$ such that*

$$|1 - f_i(t)| \le Tf(t), \quad |f_i(\pm t)| \le f(t), \quad \text{and} \int_\Gamma f(t)^m dt \le \phi(m), \tag{4}$$

for all $i, m \ge 0$, and $t \in \Gamma$. Then, for some constant $K = c(\alpha, d, T)$

$$\mathrm{var}(L_n(\alpha)) \le Kn\Big(\sum_{i=0}^{n-1}\phi([i/2])\Big)^{2\alpha-4}\sum_{j=0}^{n} j\phi([j/2])\sum_{k=j}^{2n}\phi([k/2]) =: \Delta_n(\alpha,\phi).$$

Proof of Proposition 6. Using the notation of Proposition 4, for positive integers a, u, b, v, with $a + u \le b$, $\epsilon_j = \pm 1$ and any $x \in \mathbb{Z}^d$

$$\mathbb{P}\Big(S_{a,u} + (\epsilon, S_{b,v}) = x\Big) \le \frac{1}{(2\pi)^d}\int_\Gamma \prod |f_j(\epsilon_j t)| \mathrm{d}t \le \frac{1}{(2\pi)^d}\int_\Gamma f(t)^{u+v}\mathrm{d}t \le \frac{1}{(2\pi)^d}\phi(u+v)$$

To find $\psi(u, v)$, notice that since $f(t) \ge 0$,

$$\phi(m) \ge \int_\Gamma f(t)^m[1 - f(t)^m]\mathrm{d}t = \sum_{j=0}^{m-1}\int_\Gamma f(t)^{m+j}(1-f(t))\mathrm{d}t \ge m\int_\Gamma f(t)^{2m}(1-f(t))\mathrm{d}t =: Q(2m)$$

whence $Q(m) \le \phi([m/2])/m$, where $[\cdot]$ denotes integer part. Therefore,

$$\mathbb{P}(S_{a,u} = 0) - \mathbb{P}(S_{a,u} + S_{b,1} = 0) \le CT\int_\Gamma f(t)^u(1 - f(t))dt \le CT\phi([u/2])/u,$$

and it easily follows that (B) is satisfied with $\psi(u, v) := \phi([u/2])\min(u, v)/u$. Thus all conditions of Proposition 4 are satisfied and the result follows from direct application of (4). □

The following Corollary, allows for the case where $\phi(m)$ is regularly varying.

Corollary 7. *Assume that the conditions of Proposition 6 are satisfied with $\phi(m) = h(m)m^{-r}$, $r \ge 1$, where $h(x)$ is a slowly varying at $x \to \infty$. Then,*

$$\mathrm{var}(L_n(\alpha)) \le K\Delta_n(\alpha,\phi) \le c_\alpha T^{2\alpha-2}\begin{cases} n^2\Big[\sum_{k=1}^n \frac{h(k)}{k}\Big]^{2\alpha-4}, & \text{for } r = 1,\\ n^{4-2r}h^2(n), & \text{for } 1 < r < 3/2,\\ n\sum_{k=1}^n h(k)^2/k, & \text{for } r = 3/2, \text{ and}\\ n, & \text{for } r > 3/2.\end{cases}$$

Again, the cited relevant results treated so far are not only obtained as a special case but also extended to dependent variables such as a random walk on a hidden Markov chain. In addition, following Kesten and Spitzer [13] we can mimic the behaviour of genuinely d-dimensional random walk by constructing a one dimensional symmetric random walk with characteristic function $f(t) = 1 - c|t|^{1/r} + o(|t|^{1/r})$ with $r = 2/d$ for $d = 2, 3$ and $r = 1/2$ for $d \ge 4$.

The following example of genuinely 2-dimensional recurrent walk with infinite variance was motivated by Spitzer [19, pp. 87].

Example 8. Let $S_n = \sum_{i=1}^n X_i$ be a random walk in $\mathbb{Z}^2$, such that $\mathbb{P}(|X| = k) = c/(k^3\log(k)^\gamma)$, for $k \ge 4$ and $\gamma \in [0, 1)$. Then we have $\mathrm{var}(L_n(\alpha)) \le cn^2\max\{[\log n]^\gamma, \log\log n\}^{2\alpha-4}\log n^{-2(1-\gamma)}$, for $n \ge 10$. Under these assumptions we have $\mathbb{P}(S_n = 0) \le c/n\log(n)^{1-\gamma}$, which is in the *critical range*, where the random walk is recurrent, without second moment. To show it, we notice that by lengthy straightforward calculation the characteristic function of X satisfies (4) with

$$\phi(n) = \frac{c}{n\log(\mathrm{e}\vee n)^{1-\gamma}}, \ f(t) = \exp[-A|t|^2h(|t|^2)], \quad \text{where} \quad h(r) := \big[1 + \log(1/r)_+\big]^{1-\gamma},$$

and the sequence $\phi(m)$ is identified via Fourier inversion, polar coordinates and a Laplace argument

$$\int_\Gamma f(t)^n dt \le c\int_0^1 \exp\big[-nr(1 + \log(1/r))^{1-\gamma}\big] + O(\mathrm{e}^{-n}) \le \frac{c}{n\log(\mathrm{e}\vee n)^{1-\gamma}} =: \phi(n).$$

2.2. Bounds for identically distributed variables.

Proposition 9 (General upper bound for i.i.d.)**.** *Let X_i be i.i.d. $\mathbb{Z}^d$-valued random variables, and suppose that for all $n \in \mathbb{N}$, positive integers a, u, b, v, with $a + u \leq b$, and any $x \in \mathbb{Z}^d$*

$$\mathbb{P}\Big(S_{a,u} \pm S_{b,v} = x\Big) \leq \phi(u+v), \tag{5}$$

where $\phi(m)$ is a non-increasing sequence. Then, for some constant $K = c(\alpha)$

$$\operatorname{var}(L_n(\alpha)) \leq Kn\Big(\sum_{i=0}^{n-1} \phi(i)\Big)^{2\alpha-4} \sum_{j=0}^{n} j\phi(j) \sum_{k=j}^{[\alpha n]+1} \phi([k/\alpha]) \ .$$

Proof of Proposition 9. By inspecting the proof of Proposition 6, we notice that only need to bound the J_n term. Consider a typical ordering

$$0 \leq i_1 \leq \cdots \leq i_k \leq j_1 \leq \cdots \leq j_\alpha \leq i_{k+1} \leq \cdots \leq i_\alpha \leq n.$$

let us change variables to $(m_0, \dots, m_{2\alpha})$ such that $m_0 + \cdots + m_{2\alpha} = n$. Then the contribution from this case to J_n is

$$\sum_{m_0,\dots,m_{2\alpha}} \prod_{\substack{j \neq k, k+\alpha \\ 1 \leq j \leq 2\alpha-1}} \mathbb{P}(S_{m_j} = 0)\Big[\mathbb{P}(S_{m_k+m_{k+\alpha}} = 0) - \mathbb{P}(S_{m_k+\cdots+m_{k+\alpha}} = 0)\Big]. \tag{6}$$

For $j \neq \alpha, k+\alpha$ keep m_j fixed and sum over $m = m_k + m_{k+\alpha}$, from 0 to M which depends on n, and the m_j for $j \neq k, k+\alpha$. Then for given $m_{k+1}, \dots, m_{k+\alpha-1}$, the term in the sum is

$$\sum_{m=0}^{M} (m+1)[\mathbb{P}(S_m = 0) - \mathbb{P}(S_{m+q} = 0)],$$

where $q := m_{k+1} + \ldots + m_{k+\alpha-1}$. Then since $M \leq n - q$, it is an easy exercise to show that this sum is bounded above by

$$\begin{aligned} &\sum_{m=0}^{M} (m+1)\Big[\mathbb{P}(S_m = 0) - \mathbb{P}(S_{m+q} = 0)\Big] \\ &\leq \sum_{m=0}^{q-1} (m+1)\mathbb{P}(S_m = 0) + q\mathbf{I}(n-q \geq q) \sum_{m=q}^{n-q} \mathbb{P}(S_m = 0) \\ &\leq \sum_{m=0}^{(\alpha m^*)\wedge n} (m+1)\mathbb{P}(S_m = 0) + \alpha m^* \sum_{m=m^*}^{n} P(S_m = 0) \end{aligned}$$

where $m^* := \max\{m_{k+1}, \dots, m_{k+\alpha-1}\}$. The result follows by summing over all indices apart from m^* and changing the order of summation. □

2.3. Proofs of main results.

Proof of Theorem 1. We apply a comparison argument found to be useful in many areas (e.g. Pruss and Montgomery-Smith [18], and Lefevre and Utev [16]), more exactly, we bound $\operatorname{var}(L_n)$ by the corresponding characteristic for the symmetrised random walk.

Following Spitzer's argument we notice that with $f(t) = \mathbb{E}[\exp(it \cdot X_1)]$

$$\mathbb{P}\Big(S_{a,u} + \epsilon S_{b,v} = x\Big) \leq c\int_\Gamma |f(t)|^u |f(-t)|^v \mathrm{d}t = c\int_\Gamma \Big[|f(t)|^2\Big]^{u/2} \Big[|f(-t)|^2\Big]^{v/2} \mathrm{d}t$$

Since $|f(t)|^2$ is a characteristic function of d-dimensional symmetric integer variable, for some positive λ, $1 - |f(t)|^2 \geq \lambda |t|^2$, and hence,

$$\mathbb{P}\Big(S_{a,u} + \epsilon S_{b,v} = x\Big) \leq c\int_\Gamma \exp\Big[-\frac{\lambda(u+v)}{2}|t|^2\Big]\mathrm{d}t \leq c(u+v)^{-d/2}$$

and the proof follows from Proposition 9 applied with $\phi(m) = m^{-d/2}$. □

The proof of Theorem 2 will be based on the following Lemma.

Lemma 10. *Assume X is genuinely d-dimensional and $\mathbb{E}|X|^2 = \infty$. Then there exists a monotone slowly varying function $h_n \to 0$ as $n \to \infty$ such that*

$$\sup_{x\in\mathbb{Z}^d} \mathbb{P}(S_n = x) \le c_d \int_\Gamma |\mathbb{E}e^{it\cdot X}|^n dt \le h_n n^{-d/2}$$

Proof of lemma 10. Without loss of generality assume that X is symmetrized. Let $\sigma_{e,L} := \mathbb{E}[(e\cdot X)^2 \mathbf{I}(|X| \le L)]$. Following Spitzer, since X is genuinely d-dimensional, we may assume that there exist positive constants c and W such that for any unit vector $|e| = 1$, $\sigma_{e,W} \ge c$ and $1 - f(t) \ge c|t|^2$. Let λ_d be the d-dimensional Lebesgue measure on $\mathbb{R}^d$, and μ_d the Lebesgue-Haar measure on $S^{d-1} := \{e \in [-\pi,\pi]^d : |e| = 1\}$. Notice that since $\mathbb{E}|X|^2 = \infty$, for any K we have $\mu_d\{e : \sigma_{e,\infty} < K\} = 0$.

Fix a small positive x such that $\sqrt{c/x} \ge 2W$, and for any $\epsilon > 0$ let $K = K(\epsilon) = \epsilon^{-d/2}$. Then there exists $L = L(\epsilon) > 0$ small enough so that $\mu_d\{e : \sigma_{e,L} < K\} \le \epsilon^{d/2}$. We partition S^{d-1} in two sets

$$A_{L,K} = \{e \in S^{d-1} : \sigma_{e,L} \ge K\} \quad\text{and}\quad \bar{A}_{L,K} = \{e \in S^{d-1} : \sigma_{e,L} < K\}\ ,$$

so that for any direction $e \in \bar{A}_{L,K}$,

$$\{z \in \mathbb{R} : 1 - f(ze) \le x\} \subseteq \{z : cz^2 \le x\} \subseteq \{z : |z| \le \sqrt{x/c}\}\ .$$

Hence, using d-dimensional spherical coordinates,

$$\lambda_d\{(z,e) \in \mathbb{R}\times\bar{A}_{L,K} : 1 - f(ez) \le x\} \le \mu_d\{\bar{A}_{L,K}\}(x/c)^{d/2}(1/d) \le \epsilon^{d/2}(x/c)^{d/2}(1/d)\ .$$

On the other hand, for any t,

$$1 - f(t) = 2\sum_{k\in Z^d} \sin([t\cdot k]/2)^2 P(X = k) \ge (1/4)E[(t\cdot X)^2 I(|t\cdot X| \le 1/2)] = (|t|^2/4)\sigma_{t/|t|,1/2|t|}.$$

Now, assume that $\sqrt{c/x} \ge 2L$. Then for any direction $e \in A_{L,K}$, by choice of x and since $\sigma_{e,L}$ is increasing in L, for $cz^2 \le 1 - f(ez) \le x$ or $|z| \le \sqrt{x/c}$, it must be the case that

$$x \ge 1 - f(ez) \ge (z^2/4)\sigma_{e,1/2z} \ge (z^2/4)\sigma_{e,L} \ge (z^2/4)K$$

implying that on the set $A_{L,K}$, it must be that $|z| \le 2\sqrt{x/K}$. Changing to d-dimensional polar coordinates, we find that

$$\lambda_d\Big\{(z,e) \in \mathbb{R}\times A_{L,K} : 1 - f(ez) \le x\Big\} \le \int_{A_{L,K}}\int_0^{\sqrt{4x/K}} r^{d-1}\mathrm{d}r\mathrm{d}e \le C_d\epsilon^{d/2}x^{d/2}\ .$$

Overall, for $x \le c/4L^2$, $\lambda_d\{t : 1 - f(t) \le x\} \le c_d(x\epsilon)^{d/2}$, and hence $\{t \in \Gamma : 1 - f(t) \le x\}$ has Lebesgue measure $o(x^{d/2})$.

Let $F(x)$ be the cumulative distribution function of $\log(1/f(t))$ on the probability space Γ with normalised Lebesgue measure. Then F is continuous at $x = 0$ and supported on $\mathbb{R}^+$. Moreover, as $0 < x \to 0$, $F(x) = o(x^{d/2})$. Therefore, for some positive sequence $\epsilon_n \to 0$

$$\frac{1}{[2\pi]^2}\int_\Gamma f(t)^n \mathrm{d}t = \int_0^\infty \mathrm{e}^{-nx}\mathrm{d}F(x) = n\int_0^\infty \mathrm{e}^{-nx}F(x)\mathrm{d}x \le n^{-d/2}\epsilon_n.$$

It remains to show that there exists a positive monotone slowly varying function $\epsilon_n \le h(n) \to 0$ as $n \to \infty$. Let $\delta_n = \sup_{j\ge n}\epsilon_j$, $a_0 := 0$ and for $n \ge 1$ define a_n recursively by $a_n = \min(2a_{2^{r-1}}, 1/\delta_n)$, for $2^{r-1} < n \le 2^r$, so that $a_n \to \infty$ is monotone, $a_{2^r} \le 2a_{2^{r-1}}$ implying that $a_{2n} \le 4a_n$, and $1/a_n \ge \delta_n \ge \epsilon_n$. Finally, take $h_n := 1/\max(a_0, \log a_n)$. □

Proof of Theorem 2. Assume that $\mathbb{E}|X|^2 = \infty$ and $d = 2$ or $d = 3$. Then, by Lemma 10 there exists a slowly varying function $h(n) \to 0$ as $n \to \infty$ such that $\int_\Gamma |\mathbb{E}\exp(it\cdot X)|^n\mathrm{d}t \le h_n n^{-d/2}$. Applying Corollary 7 with $r = 1$ and $r = 3/2$ we respectively find that

$$\operatorname{var}(L_n(\alpha)) \le \begin{cases} Kn^2\Big(\sum_{k=1}^n h(k)/k\Big)^{2\alpha-4} = o(n^2(\log n)^{2\alpha-4}), & \text{for } d = 2, \text{ and}\\ Kn\Big(\sum_{k=1}^n h(k)^2/k\Big) = o(n\ln n), & \text{for } d = 3.\end{cases}$$

Finally assume that $\mathbb{E}|X|^2 < \infty$ and $E[X] = \mu \neq 0$. Then $\mathbb{P}(S_n = 0) = \mathbb{P}(S'_n = -n\mu)$ whence it follows that $\mathbb{P}(S_n = 0) = o(n^{-d/2})$ (see for example [15, Theorem 2.3.10]). Then inspecting the proof of Proposition 4, one can readily obtain the desired bound for the J_n term, while with slight modification the bound for the I_n term also follows.

Note that for $d = 1$ the situation is much simpler since then $\mathrm{var}(L_n(\alpha|\epsilon, d)) \sim C[\mathbb{E}L_n(\alpha|\epsilon, d)]^2$ and if $\mathbb{E}|X|^2 = \infty$ or $\mathbb{E}[X] \neq 0$, $\mathbb{E}L_n(\alpha|\epsilon, d) = o(n^{(1+\alpha)/2})$. □

Proof of Theorem 3. We first give the proof for the case $d = 1$. As in the proof of Proposition 4 we begin from expression (2), and define the sequences p_i, and δ_i for $i = 1, \ldots, 2\alpha - 1$, and the quantity $v(\delta) = \sum_{i=1}^{2\alpha-1} |\delta_i|$. Recall that $v(\delta)$ measures the interlacement of the two sequences $k_1, \ldots, k_\alpha$, and $l_1, \ldots, l_\alpha$. For example $v(\delta) = 1$ occurs when either $k_\alpha \leq l_1$, or $l_\alpha \leq k_1$, in which case the contribution vanishes by the Markov property. On the other hand $v(\delta) = 2$ when for example $l_1, \ldots, l_\alpha \in [k_i, k_{i+1}]$ for some i. Finally $v(\delta) = 3$ occurs when for example

$$k_1 \leq \cdots \leq k_r \leq l_1 \leq \cdots \leq l_s \leq k_{r+1} \leq \cdots \leq k_\alpha \leq l_{s+1} \leq \cdots \leq l_\alpha \leq n.$$

From the proof of Proposition 4, and using the bound $\mathbb{P}(S_n = 0) = O(1/n)$, the terms of the sum are bounded above by $n^2 \log(n)^{2\alpha-1-v(\delta)}$, and thus the leading term appears when either $v(\delta) = 2, 3$, with other terms giving strictly lower order. We shall therefore analyze these two situations in detail in order to derive the exact asymptotic constants. When $v = 3$, the two terms in the difference individually give the correct order and shall be treated by the classical Tauberian theory. However for $v = 2$, the two terms only give the correct order when considered together. This however forbids the use of Karamata's Tauberian theorem since the monotonicity restriction would require roughly that X_i is symmetrized. Thus the complex Tauberian approach, as developed in [10], is required to justify the answer.

Case 1: $v(\delta) = 3$. Assume that part of the sequence $\mathbf{l} = \{l_1, \ldots, l_\alpha\}$ lies between k_r and k_{r+1}, and the rest between k_s and k_{s+1}. Then using the change of variables

$$\begin{gathered}
i_1 = m_0, i_2 = m_0 + m_1, \cdots, i_r = m_0 + \cdots + m_{r-1} \\
j_1 = m_0 + \cdots + m_r, j_2 = m_0 + \cdots + m_{r+1}, \cdots, j_s = m_0 + \cdots + m_{r+s-1}, \\
i_{r+1} = m_0 + \cdots + m_{r+s}, i_{r+2} = m_0 + \cdots + m_{r+s+1}, \cdots, i_\alpha = m_0 + \cdots + m_{\alpha+s-1} \\
j_{s+1} = m_0 + \cdots + m_{\alpha+s}, j_{s+2} = m_0 + \cdots + m_{\alpha+s+1}, \cdots, j_\alpha = m_{2\alpha-1}, n = m_0 + \cdots + m_{2\alpha}.
\end{gathered}$$

we rewrite the positive term in (2) as

$$\begin{aligned}
a(n) &= \sum \mathbb{P}\Big[S(i_1) = \cdots = S(i_\alpha); S(j_1) = \cdots = S(j_\alpha)\Big] \\
&= \sum_{m_0, \cdots, m_{2\alpha-1}} \Big[\prod_{\substack{j=1 \\ j \neq r, r+s, \alpha+s}}^{2\alpha-1} \mathbb{P}(S_{m_j} = 0)\Big] \times \mathbb{P}(S_{m_r} + S'_{m_{r+s}} = S'_{m_{r+s}} + S''_{m_{\alpha+s}} = 0).
\end{aligned}$$

Notice that from [10] we have that $\sum_{n\geq 0} \lambda^n \mathbb{P}(S_n = 0) \sim \log\left(1/(1-\lambda)\right)/\pi\gamma$. Let

$$a(\lambda) = (1-\lambda)^{-3}[-\log(1-\lambda)]^{2\alpha-4}\ , \ c_\gamma = (\pi\gamma)^{-2\alpha+4}.$$

Then, by direct calculations and Fourier inversion formula

$$\begin{aligned}
\sum_{n\geq 0} \lambda^n a(n) &= c_\gamma (1-\lambda) a(\lambda) \sum_{x\in\mathbb{Z}} \sum_{k_1,k_2,k_3\geq 0} \lambda^{k_1+k_2+k_3} \mathbb{P}(S_{k_1} = x)\mathbb{P}(S_{k_2} = -x)\mathbb{P}(S_{k_3} = x) \\
&= c_\gamma (1-\lambda) a(\lambda) \frac{1}{(2\pi)^2} \iint_{[-\pi,\pi]^2} \frac{\mathrm{d}t\mathrm{d}s}{(1-\lambda f(t))(1-\lambda f(s))(1-\lambda f(t+s))} \\
&\sim c_\gamma (1-\lambda) a(\lambda) \frac{1}{(2\pi)^2\gamma^2} \frac{1}{1-\lambda} \iint_{\mathbb{R}^2} \frac{\mathrm{d}x\mathrm{d}y}{(1+|x|)(1+|y|)(1+|x+y|)} \sim (1/4\gamma^2) c_\gamma a(\lambda)
\end{aligned}$$

Next we consider the negative term in (2)

$$b(n) := \sum_{m_0,\ldots,m_{2\alpha-1}} \mathbb{P}\Big[S_{m_1} = \cdots = S_{m_{r-1}} = S_{m_r} + \cdots + S_{m_{r+s}} = S_{m_{r+s+1}} = \cdots = S_{m_{\alpha+s-1}} = 0\Big]$$

$$\times\, \mathbb{P}\Big[S_{m_{r+1}} = \cdots = S_{m_{r+s}} + \cdots + S_{m_{\alpha+s}} = S_{m_{\alpha+s+1}} = \cdots = S_{m_{2\alpha-1}} = 0\Big].$$

By direct calculations and (1),

$$\sum_n \lambda^n b(n) = \Big(\frac{1}{\pi\gamma}\log\big(\frac{1}{1-\lambda}\big)\Big)^{\alpha-s+r-2}(1-\lambda)^{-2}\sum_{m_r,\ldots,m_{\alpha+s}=0}^{\infty}\lambda^{m_r+\cdots+m_{\alpha+s}}$$
$$\times \prod_{\substack{t=r+1,\cdots,\alpha+s-1\\ t\neq r+s}}\mathbb{P}(S_{m_t}=0)\mathbb{P}(S_{m_r}+\cdots+S_{m_{r+s}}=0)\mathbb{P}(S_{m_{r+s}}+\cdots S_{m_{\alpha+s}}=0),$$

and using Fourier inversion and (1) the internal sum behaves as

$$(2\pi)^{-\alpha-s+r}\int_{-\pi}^{\pi}\!\!\cdots\!\int_{-\pi}^{\pi}(1-\lambda\phi(x))^{-1}(1-\lambda\phi(x)\phi(y))^{-1}(1-\lambda\phi(y))^{-1}$$
$$\times\Big[\prod_{j=r+1}^{r+s-1}\prod_{k=r+s+1}^{\alpha+s-1}(1-\lambda\phi(x)\phi(t_j))^{-1}(1-\lambda\phi(y)\phi(t_k))^{-1}\mathrm{d}t_j\,\mathrm{d}t_k\Big]\mathrm{d}x\,\mathrm{d}y$$
$$\sim(\pi\gamma)^{-\alpha-s+r}(1-\lambda)^{-1}\log\big(\frac{1}{1-\lambda}\big)^{\alpha-r+s-2}\frac{\pi^2}{6}.$$

Then we have $\sum_n\lambda^n b(n)\sim(\pi^2/6(\pi\gamma)^{2\alpha-2})a(\lambda)$, whence the Tauberian theorem implies that $a(n)-b(n)\sim n^2\log(n)^{2\alpha-4}/24\pi^{2\alpha-4}\gamma^{2\alpha-2}$. Most importantly we see that the lengths and locations of the chains, r and s, do not affect the asymptotic. Noting that if $1\le r,s\le\alpha-1$, we can partition $2\alpha=r+s+(\alpha-r)+(\alpha-s)$ in $(\alpha-1)^2$ ways, and thus overall the total contribution from terms with $v=3$ is

$$[(\alpha!(\alpha-1))^2/12\pi^{2\alpha-4}\gamma^{2\alpha-2}]n^2\log(n)^{2\alpha-4}.$$

Case 2: $v(\delta)=2$. The typical term $c(n)$ was introduced in (6) in the proof of Proposition 9. Now we let $\lambda\in\mathbb{C}$, with $|\lambda|<1$. By lengthy but direct calculations we can derive an expression of the form

$$\sum_n\lambda^n c(n)=\frac{\alpha-1}{(\gamma\pi)^{2\alpha-2}}a(\lambda)+o(a(\lambda)),\quad \lambda\to1.$$

The approach developed in [10] can then be used to bound the error terms and show that $c(n)\sim[(\alpha-1)/2(\gamma\pi)^{2\alpha-2}]n^2\log(n)^{2\alpha-4}$.

Finally taking into account the fact that the $l_1,\ldots,l_\alpha$ can be in any of the $\alpha-1$ intervals $[k_i,k_{i+1}]$, for $i=1,\ldots,\alpha-1$, the result follows the overall contribution of terms with $v(\delta)=2$ is

$$\frac{(\alpha-1)^2}{2(\gamma\pi)^{2\alpha-2}}n^2\log(n)^{2\alpha-4}.$$

The case for $d=2$ is very similar, so we move on to the case $d=3$.

Case $d=3$, $\alpha=2$. Using the same notation as before, we have three terms to consider $a(n)$, $b(n)$, and $c(n)$. We first consider $c(n)$. Letting $K:=\epsilon/\sqrt{1-\lambda}$ and using the usual power series construction and spherical coordinates

$$\sum_n\lambda^n c(n)=(1-\lambda)^{-2}(2\pi)^{-6}\iint_{J^3\times J^3}\frac{\lambda f(y)(1-f(x))\mathrm{d}x\mathrm{d}y}{(1-\lambda f(x))^2(1-\lambda f(y))(1-\lambda f(x)f(y))}$$
$$\sim 2(2\pi)^{-4}|\Sigma|^{-1}(1-\lambda)^{-2}\int_0^K\int_0^K\frac{r^4s^2\mathrm{d}r\mathrm{d}s}{(1+r^2)^2(1+s)^2(1+r^2+s^2)}$$
$$\sim 2(2\pi)^{-4}|\Sigma|^{-1}\frac{\pi}{2}(1-\lambda)^{-2}\log\Big(\frac{1}{1-\lambda}\Big)=:\kappa_1(1-\lambda)^{-2}\log\Big(\frac{1}{1-\lambda}\Big), \tag{7}$$

and thus $c(n)\sim\kappa_1 n\log n$, where $\kappa_1>0$, where the answer can be justified following [10].

The term $a(n)-b(n)$ is trickier to compute. As usual we consider the power series

$$\sum_{n\ge0}\lambda^n(a(n)-b(n))=(1-\lambda)^{-2}(2\pi)^{-6}\iint_{B(\epsilon)}\frac{\mathrm{d}x\mathrm{d}y}{(1-\lambda f(x))(1-\lambda f(y))(1-\lambda f(x+y))}$$

$$- (1-\lambda)^{-2}(2\pi)^{-6} \iint_{B(\epsilon)} \frac{\mathrm{d}x\mathrm{d}y}{(1-\lambda f(x))(1-\lambda f(y))(1-\lambda f(x)f(y))}$$
$$= (1-\lambda)^{-2}(2\pi)^{-6}(I_1(\lambda) - I_2(\lambda)).$$

Let $A \in [-1,1]$ be the cosine of the angle between x and y, which in spherical coordinates is

$$A = A(\theta_1,\theta_2,\phi_1,\phi_2) = \cos(\phi_1-\phi_2)\sin(\theta_1)\sin(\theta_2) + \cos(\theta_1)\cos(\theta_2). \tag{8}$$

Then as $0 < \lambda \uparrow 1$, using the expansion (1)

$$I_1(\lambda) \sim |\Sigma|^{-1} \int_{r,s=0}^{\epsilon} \int_{\phi_{1,2}=0}^{2\pi} \int_{\theta_{1,2}=0}^{\pi} \frac{r^2 s^2 \sin(\theta_1)\sin(\theta_2)\mathrm{d}r\mathrm{d}s\mathrm{d}\theta_1\mathrm{d}\theta_2\mathrm{d}\phi_1\mathrm{d}\phi_2}{(1-\lambda+\lambda r^2)(1-\lambda+\lambda s^2)\Big[1-\lambda+\lambda(r^2+s^2+2Ars)\Big]}$$
$$= |\Sigma|^{-1} \int_{\phi,\theta} \sin(\theta_1)\sin(\theta_2) \int_{r=0}^{K}\int_{s=0}^{K} \frac{r^2s^2\mathrm{d}r\mathrm{d}s}{(1+r^2)(1+s^2)\Big[1+r^2+s^2+2Ars\Big]}\mathrm{d}\theta\mathrm{d}\phi$$
$$\sim |\Sigma|^{-1}\log(K)\int_{\phi,\theta}\sin(\theta_1)\sin(\theta_2)\frac{\arccos(A(\theta,\phi))}{\sqrt{1-A(\theta,\phi)^2}}.$$

The other integral is slightly easier

$$I_2(\lambda) \sim |\Sigma|^{-1}\frac{\pi}{2}\log K \int_{\theta,\phi}\sin(\theta_1)\sin(\theta_2)\mathrm{d}\theta_1\mathrm{d}\theta_2\mathrm{d}\phi_1\mathrm{d}\phi_2,$$

and thus overall we must have that

$$(I_1 - I_2)(\lambda) \sim \frac{1}{2}(2\pi)^{-6}|\Sigma|^{-1}(1-\lambda)^{-2}\log\Big(\frac{1}{1-\lambda}\Big)$$
$$\times \int_{\theta_1,\theta_2=0}^{\pi}\int_{\phi_1,\phi_2=0}^{2\pi}\Big[\frac{\arccos(A)}{\sqrt{1-A^2}} - \frac{\pi}{2}\Big]\sin(\theta_1)\sin(\theta_2)\mathrm{d}\theta_{1,2}\mathrm{d}\phi_{1,2}$$
$$=: \kappa_2(1-\lambda)^{-2}\log\Big(\frac{1}{1-\lambda}\Big), \tag{9}$$

whence it follows that $\mathrm{var}(L_n(2)) \sim (\kappa_1+\kappa_2)n\log n$.

To prove the last claim, let $S_n' = X_1' + \cdots + X_n'$ be another random walk, independent of S_n, such that its characteristic function $f'(t) = \mathbb{E}[\exp(\mathrm{i}tX_i')]$ also satisfies the expansion (1). Then using [10, Lemma 3.1] one can adapt the proof of [10, Theorem 2.1] to show that $L_n'(\alpha) = L_n(\alpha) + o(L_n(\alpha))$. □

References

[1] J. Aaronson. Relative complexity of random walks in random sceneries. *Ann. Probab.* 40, 2012, no. 6, pp. 2460-2482.

[2] A. Asselah. Shape transition under excess self-intersections for transient random walk. *Annales de l'institut Henri Poincare (B) Probability and Statistics*, 46, no. 1, pp. 1250-278, 2010.

[3] M. Becker and W. König. Moments and distribution of the local times of a transient random walk on $\mathbb{Z}^d$. *J. Theoret. Probab.*, 22 (2): 365–374, 2009.

[4] E. Bolthausen. A central limit theorem for two-dimensional random walks in random sceneries. *Ann. Probab.*, 17 (1): 108–115, 1989.

[5] A. N. Borodin. A limit theorem for sums of independent random variables defined on a recurrent random walk. *Dokl. Akad. Nauk SSSR*, 246 (4): 786–787, 1979.

[6] D. C. Brydges and G. Slade. The diffusive phase of a model of self-interacting walks. *Probab. Theory Related Fields*, 103 (3): 285–315, 1995.

[7] F. Castell, N. Guillotin-Plantard, and F. Pène. Limit theorems for one and two-dimensional random walks in random scenery. *Annales de l'institut Henri Poincare (B) Probability and Statistics*, 2012.

[8] X. Chen. *Random walk intersections*, volume 157 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2010. Large deviations and related topics.

[9] G. Deligiannidis and S. Utev. An asymptotic variance of the self-intersections of random walks, 2010, arXiv:1004.4845.

[10] G. Deligiannidis and S. Utev. Computation of the asymptotics of the variance of the number of self-intersections of stable random walks using Wiener-Darboux theory. *Sib. Math. J*, 52, 2011.

[11] G. Deligiannidis and K. Zemer. Relative complexity of random walks in random scenery in the absence of a weak invariance principle for the local times. preprint, 2015.

[12] Jürgen Gärtner and Rongfeng Sun. A quenched limit theorem for the local time of random walks on $\mathbb{Z}^2$. *Stochastic Process. Appl.*, 119 (4): 1198–1215, 2009.

[13] H. Kesten and F. Spitzer. A limit theorem related to a new class of self-similar processes. *Z. Wahrsch. verw. Gebiete*, 50: 5–25, 1979.

[14] G.F. Lawler. *Intersections of Random Walks*. Birkhauser, 1991.

[15] Gregory F. Lawler and Vlada Limic. *Random walk: a modern introduction*, volume 123 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2010.

[16] C. Lefèvre and S. Utev. Exact norms of a Stein-type operator and associated stochastic orderings. *Probab. Theory Related Fields*, 127 (3): 353–366, 2003.

[17] T. M. Lewis. A law of the iterated logarithm for random walk in random scenery with deterministic normalizers. *J. Theoret. Probab.*, 6 (2): 209–230, 1993.

[18] S. J. Montgomery-Smith and A. R. Pruss. A comparison inequality for sums of independent random variables. *J. Math. Anal. Appl.*, 254 (1): 35–42, 2001.

[19] F. Spitzer. *Principles of Random Walk* . Springer, 1976.

[20] J. Černỳ. Moments and distribution of the local time of a two-dimensional random walk. *Stochastic Process. Appl.*, 117 (2): 262–270, 2007.

Department of Statistics, University of Oxford, Oxford OX1 3TG, UK
E-mail address: deligian@stats.ox.ac.uk

Department of Mathematics, University of Leicester, LE1 7RH,UK
E-mail address: su35@le.ac.uk